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The Past of a Stopping Point and Stopping for Two-Parameter Processes

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Optional increasing paths passing through a given stopping point are studied. A characterization of the two extreme optional increasing paths is obtained. The past of a stopping point is defined, and a description of the largest stopping point smaller than two given stopping points is given. A stopping procedure is naturally associated with this notion of infimum. Stopped martingales and stopped filtrations are studied. "Local martingales" are defined and studied along horizontal and vertical lines. A nontrivial example of "local martingale" is given.

INTRODUCTION

Stopping for two-parameter processes is not an easy matter. Stochastic integrals have been used to stop L^2-bounded martingales at a stopping line (see, among others [1, 6, 18, 15]). This method is of limited use, for at least two reasons: the class of processes that we can stop by this method is restricted (and not well known); the result of this operation is not attached to the value of the process at a point.

In this article we indicate how to stop any process at a stopping point. This method is far from perfect, again, at least for two reasons: the study of two-parameter processes requires more than the notion of stopping points, and the result of this operation is not obviously related to the stochastic integral. Nevertheless, we prove encouraging results concerning the stopped filtration and stopped martingales. Our main tool is the notion of optional increasing path (OIP), introduced by Walsh [17].

In Section I we define the optional increasing paths as we did in [9], and we prove (Theorem 3) that, under the conditional independence (F_4), there exists an OIP "passing through" any given stopping point. In [9] we studied

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the optimal stopping problem along such paths; Mazziotto and Szpirglas [12] solved this problem in the plane (at least in the discrete plane).

In Section II we note that the previous construction gives us the two extreme OIP passing through a given stopping point. We define the past of a stopping point as the random set lying between these two OIP, and not in the future of the stopping point. Propositions 4 and 6 justify this definition.

In Section III we define the infimum of two stopping points as the largest stopping point smaller than both of them; existence and unicity are consequences of the Section II. This has already been used by Merzbach [14]. Proposition 9 characterizes the σ-algebra associated with the infimum of a stopping point and a deterministic point.

In Section IV we indicate how to stop any process at a stopping point by replacing the geometrical infimum by the infimum defined in Section III. Proposition 11 tells us that a stopped martingale is a martingale with respect to the stopped filtration; this allows us to describe the σ-algebra associated with a stopping point of the stopped filtration. Proposition 13 shows that the conditional independence (F_4) is preserved by stopping. Theorem 14 answers the question about a stopped martingale with respect to the initial filtration; in general, this is not a martingale but a weak martingale.

As pointed out in [15], localization for two-parameter processes is not an easy problem. In Section V we do not intend to solve it, but rather to show how our stopping method leads to a localization procedure. Proposition 16 is an encouraging result about the behavior of a "local martingale" along horizontal or vertical lines. Finally an example of "local martingale" is given.

The discrete version of our results is an easy adaptation (for instance, see [9] for the discrete version of Theorem 3).

The results of this article were, in part, announced in our note [10]. We like to take this opportunity to mention a mistake in this note: Theorem 5 of [10] has to be replaced by Proposition 11 and Theorem 14 of this article.

NOTATIONS

The index set is $\mathbb{R}_+^2$ and $z=(s,t)\leqslant z'=(s',t')$ if and only if $s\leqslant s'$ and $t\leqslant t'$ ($z<z' \Leftrightarrow s<s'$ and $t<t'$). $\mathbb{R}_+^2\cup\{\infty\}$ is the one-point compactification of $\mathbb{R}_+^2$. Let R_z^i ($i=1,2,3$ or 4) denote the rectangle: $R_z^1=\{z': z\leqslant z'\}=[z,\infty[$, $R_z^2=\{z': s'\leqslant s, t'\geqslant t\}$, $R_z^3=\{z': z'\leqslant z\}=[0,z]=R_z$ or $R_z^4=\{z': s'\geqslant s, t'\leqslant t\}$. R_{z+0}^4, for instance, denotes $\{z': s'>s$ and $t'\leqslant t\}$ and R_{z+-}^4 denotes $\{z': s'>s$ and $t'<t\}$.

A subset A of $\mathbb{R}_+^2$ is *of type* i ($i=1,2,3$ or 4) if: $z\in A$ implies $R_z^i\subset A$; the empty set $\varnothing$ if of type i ($i=1,2,3$ and 4).

Let $(\Omega,\mathscr{F},P)$ be a complete probability space and let $(\mathscr{F}_z)_{z\in\mathbb{R}_+^2}$ be an increasing family of sub $-\sigma-$ algebras of $\mathscr{F}$. For every $z=(s,t)$, we set $\mathscr{F}_z^1=V_t\mathscr{F}_{s,t}=\mathscr{F}_{s,\infty}=\mathscr{F}_s^1$, $\mathscr{F}_z^2=V_s\mathscr{F}_{s,t}=\mathscr{F}_{\infty,t}=\mathscr{F}_t^2$ and $\mathscr{F}_\infty=V_z\mathscr{F}_z$. We suppose $\mathscr{F}_0$ complete and also that the filtrations $(\mathscr{F}_s^1)_{s\geqslant 0}$ and $(\mathscr{F}_t^2)_{t\geqslant 0}$ satisfy the usual conditions (completion and right continuity); moreover we suppose that the *conditional independence* (F_4 of [5]) holds: for every $z\in\mathbb{R}_+^2$, $\mathscr{F}_z^1$ and $\mathscr{F}_z^2$ are conditionally independent given $\mathscr{F}_z$.

A process $(X_z)_{z\in\mathbb{R}_+^2}$ is $(\mathscr{F}_z)$-adapted if X_z is $\mathscr{F}_z$-measurable for every z. A random set A is $(\mathscr{F}_z)$-adapted if I_A (indicator of A) is an $(\mathscr{F}_z)$-adapted process or, equivalently, if $\{\omega\in\Omega: z\in A(\omega)\}\in\mathscr{F}_z$ for every $z\in\mathbb{R}_+^2$, where $A(\omega)$ is the section of A. A random set is of type i if almost every section is of type i.

A process $(X_z)_{z\in\mathbb{R}_+^2}$ is *progressive* if for every $z'\in\mathbb{R}_+^2$, the map $(z,\omega)\to X_z(\omega)$ from $([0,z']\times\Omega, B[0,z']\otimes\mathscr{F}_{z'})$ to $(R,B(\mathbb{R}))$ is measurable (B means boreliens of). A random set A is *progressive* if the process I_A is progressive. We refer to [2] for the following notions: right limited, right continuous process; 2(or $-+$) limited, 2-continuous process; left (or 3) limited, left continuous process; 4(or $+-$) limited, 4-continuous process; the *optional* σ-algebra $\mathscr{O}$; the 1-*predictable* 2-*optional* σ-algebra $\mathscr{P}_1\cap\mathscr{O}_2$; the *predictable* σ-algebra $\mathscr{P}$; the 1-optional 2-predictable σ-algebra $\mathscr{O}_1\cap\mathscr{P}_2$. We recall that $\mathscr{P}=\mathscr{P}_1\cap\mathscr{P}_2$ and $\mathscr{O}=\mathscr{O}_1\cap\mathscr{O}_2$.

An $(\mathscr{F}_z)$-stopping point or, simply a stopping point, is a map: $Z:\Omega\to\mathbb{R}_+^2\cup\{\infty\}$ such that $\{Z\leqslant z\}\in\mathscr{F}_z$ for every $z\in\mathbb{R}_+^2$. If (S,T) denotes the random coordinates of Z, S is an $(\mathscr{F}_s^1)$-stopping time and T is an $(\mathscr{F}_t^2)$-stopping time. If $Z=(S,T)$ is an $(\mathscr{F}_z^1)$-stopping point, then S is an $(\mathscr{F}_s^1)$-stopping time and T is $(\mathscr{F}_S^1)$-measurable. The σ-algebra associated with a stopping point Z is defined by $\mathscr{F}_Z=\{A\in\mathscr{F}_\infty/A\cap\{Z\leqslant z\}\in\mathscr{F}_z, \forall z\in\mathbb{R}_+^2\}$. The notation R_z^i extends to R_Z^i for Z a stopping point.

I. Optional Increasing Paths and Stopping Points

One can define optional increasing paths as a continuously increasing family of stopping points, parametrized by $\mathbb{R}_+$ (Walsh [17]), or as the upper left boundary of a progressive set of type 4 (Fouque [9]). It turned out that these two characterizations are equivalent (see [17, 9, 14]). For the purpose of this article we do not need a parametrization and we adopt the second point of view. In [9], we extended the notion of tactic from the discrete case [11] to the continuous case, in order to get a theorem of representation of every stopping points by tactics, as in [11]. The idea of the proof of this theorem is strongly motivated by the notion of tactic. We will restrict ourself to optional increasing paths starting from the origin (as pointed out in [17], this is not a great restriction).

The next formula tells us how to obtain any optional increasing path from the progressive sets of type 4; the last two terms in this formula make the optional increasing path start from the origin.

DEFINITION 1. An *optional increasing path* Γ is a random set $(\bar{\bar{H}} \cap \overline{H^c}) \cup [(0x \times \Omega) \cap \overline{H^c}] \cup [0y \times \Omega) \cap \bar{H}]$, where H is a progressive set of type 4.

Remarks. (1) OIP will stand for optional increasing path (CCO in french!).

(2) Since the closure of a progressive set of type i ($i = 1, 2, 3$ or 4) is a progressive set of type i, Γ is a progressive set.

(3) For each $\omega, \Gamma(\omega)$ is a continuous increasing path, starting from the origin.

(4) Γ is also the "$\wedge$-boundary" of $H \cup (0x \times \Omega)$, where $z \wedge z'$ means: $s \leqslant s'$ and $t \geqslant t'$.

(5) We denote $\vec{\Gamma} = \{(\omega, z) \in \Omega \times \mathbb{R}^2_+ / \exists\, z' \in \Gamma(\omega), z' \wedge z\}$ and $\overleftarrow{\Gamma} = \{(\omega, z) \in \Omega \times \mathbb{R}^2_+ / \exists\, z' \in \Gamma(\omega),\ z \wedge z'\}$. $\vec{\Gamma}$ is a closed progressive set of type 4 and $\overleftarrow{\Gamma}$ is a closed progressive set of type 2; we have $\Gamma = \vec{\Gamma} \cap \overleftarrow{\Gamma}$.

(6) Since $I_{\vec{\Gamma}}$ is adapted and 4-continuous, $\vec{\Gamma} \in \mathscr{P}_1 \cap \mathscr{O}_2$. Therefore $\Gamma \in (\mathscr{O}_1 \cap \mathscr{P}_2) \vee (\mathscr{P}_1 \cap \mathscr{O}_2) \subset \mathscr{O}$; *$\Gamma$ is optional.*

PROPOSITION 2. *Let Γ be an optional increasing path and D a progressive set; then the random point Z, defined by $Z(\omega) = \inf\{z \in \mathbb{R}^2_+ / z \in \Gamma(\omega) \cap D(\omega)\}$, $\inf \varnothing = \infty$, is a stopping point; we call it* the entrance point of Γ in D.

Proof. For each ω, since $\Gamma(\omega) \cap D(\omega)$ is totally ordered, $Z(\omega)$ is well defined. $\{Z \leqslant z\} = \bigcap_{n \in \mathbb{N}^*} \downarrow \{[0, z + (1/n, 1/n)] \cap \Gamma(\omega) \cap D(\omega) \neq \varnothing\} \in \mathscr{F}_z$ since $\Gamma \cap D$ is progressive and $(\mathscr{F}_z)$ is right continuous.

The next result is due to Krengel–Sucheston [11] in the discrete case (Index set $= \mathbb{N}^2$), under the condition CQI (conditional qualitative independence). In the continuous case (Index set $= \mathbb{R}^2_+$), it is due to Walsh [17] or Fouque [9], under the condition CI (conditional independence or F_4).

THEOREM 3. *Let Z be a stopping point; there exists an optional increasing path Γ such that $[\![Z]\!] \subset \Gamma$ (we say that Γ* passes through Z).

Proof. For each $z = (s, t) \in \mathbb{R}^2_+$, we define almost surely the random variable $X(z) = E(I_{\{T \leqslant t\} \cap \{S > s\}} / \mathscr{F}_z) \cdot (Z = (S, T))$.

The intuitive idea of the proof is a "tactic method": if $X(z) = 0$ we *decide* "to go up" and if $X(z) > 0$ we *have* "to go to the right" (note that $\{T \leqslant t\} \cap$

$\{S > s\} = \{Z \in R^4_{z+0}\}$). First of all we have to prove the existence of a good version of X; for that, we will use an Amart theorem [16].

Let (σ, τ) be a simple 1-stopping point, D_1 and D_2, two finite subsets of $\mathbb{R}$ such that the range of (σ, τ) is included in $D_1 \times D_2 = D$.

Using the *conditional independence* and the fact that: $\{T \leqslant t\} \cap \{S > s\} \in \mathscr{F}^2_t$, we have

$$\begin{aligned}
X(\sigma, \tau) &= \sum_{(u,v)\in D} X(u, v)\, I_{\{(\sigma,\tau)=(u,v)\}} \\
&= \sum_{(u,v)\in D} E(I_{\{T\leqslant v\}\cap\{S>u\}}/\mathscr{F}^1_u) \cdot I_{\{(\sigma,\tau)=(u,v)\}} \\
&= \sum_{(u,v)\in D} E(I_{\{T\leqslant \tau\}\cap\{S>\sigma\}}/\mathscr{F}^1_u) \cdot I_{\{(\sigma,\tau)=(u,v)\}} \\
&= \sum_{u\in D_1} \left[E(I_{\{T\leqslant \tau\}\cap\{S>\sigma\}}/\mathscr{F}^1_u) \cdot I_{\{\sigma=u\}} \cdot \sum_{v\in D_2} I_{\{\tau=v\}}\right] \\
&= \sum_{u\in D_1} E(I_{\{T\leqslant \tau\}\cap\{S>\sigma\}}/\mathscr{F}^1_u) \cdot I_{\{\sigma=u\}} \\
&= E(I_{\{T\leqslant \tau\}\cap\{S>\sigma\}}/\mathscr{F}^1_\sigma).
\end{aligned}$$

Therefore $E(X(\sigma, \tau)) = P(\{T \leqslant \tau\} \cap \{S > \sigma\})$. Let us consider first, the process X defined on $\mathbb{Q}^2_+$ ($\mathbb{Q}$ for rational) by $X_z = X(z)$. The previous computation shows that, for each sequence (σ_n, τ_n) of simple 1-stopping points, taking values in $\mathbb{Q}^2_+$, decreasing in R^1, $E(X_{\sigma_n,\tau_n}) = E(X(\sigma_n, \tau_n))$ converges. Then, we can define X on $\mathbb{R}^2_+$, as the process of the right limits (we still call it X). Now, if (σ_n, τ_n) recalls (σ, τ) in R^1 [resp. in R^4], $E(X_{\sigma_n,\tau_n})$ converges to $E(X_{\sigma,\tau})$ [resp. $P(\{T < \tau\} \cap \{S > \sigma\})$]. On the other hand, $(X_{s,t}, \mathscr{F}_{s,t}, t \geqslant 0)$ is a descending amart as an increasing (bounded, right continuous) process; Corollary 2.6 [11] tells us that the 1. descending amart X has a right continuous, 4-limited modification (we still call it X).

In fact $X_z = I_{\{S>s\}} \cdot M_z$, where $M_z = E(I_{\{T\leqslant t\}}/\mathscr{F}^1_s)$ is right continuous and 4-limited. We defined X^{+-} as the process of the 4-limits of X, except for $t = 0$, where we set $X^{+-}_{s,0} = I_{\{S>s\}} E(I_{\{T<0\}}/\mathscr{F}^1_s) = 0$. Then $X^{+-}_z = I_{\{S>s\}} \cdot M^{+-}_z$, where $E(I_{\{T<t\}}/\mathscr{F}^1_s) = M^{+-}_z$ is 4-continuous and for each t, $(M^{+-}_{s,t}, \mathscr{F}^1_s, s \geqslant 0)$ is a right continuous martingale.

Using the facts that $(M_{s,t}, \mathscr{F}^1_s, s \geqslant 0)$ is a nonnegative martingale for each t, $(M_{s,t}, t \geqslant 0)$ is increasing and the factor $I_{\{S>s\}}$, we conclude that $\{X = 0\}$ is of type 4, progressive because X is right continuous, $R^2_{Z0-} \cap \{X = 0\} = \varnothing$ and $R^4_Z \subset \{X = 0\}$: therefore $[\![Z]\!] \subset \Gamma$, where Γ is the OIP obtained from $H = \{X = 0\}$.

We have $\vec{\Gamma} = \{X^{+-} = 0\}$ and $[\![Z, \infty[\![\subset \vec{\Gamma}$. Z is also the entrance point of

Γ in $[\![Z, \infty[\![$. We will denote $\underline{\Gamma_Z}$ the OIP that we constructed. Symmetrically, starting with $Y(z) = E(I_{\{S \leqslant s\} \cap \{T > t\}} / \mathscr{F}_z)$, we construct another OIP passing through Z, denoted $\underline{\gamma_Z}$ ($\bar{\gamma}_Z = \{Y^{-+} = 0\}$, where $Y_z^{-+} = I_{\{T > t\}} E(I_{\{S < s\}} / \mathscr{F}_t^2)$). We conclude this section with two remarks:

(1) $\underline{\mathscr{O}_1 \cap \mathscr{P}_2}$ (resp. $\mathscr{P}_1 \cap \mathscr{O}_2$) is generated by the evanescent sets and the sets $\vec{\Gamma}$ (resp. $\bar{\Gamma}$), where Γ is any OIP. We look at the stochastic rectangles generating these σ-algebras [2, Theorem 10] and we apply the previous theorem.

(2) We define the progressive sets $D_u = \Omega \times \{z = (s, t) \in \mathbb{R}_+^2 / s + t = u\}$ for each nonnegative real number u. Let Z_u be the entrance point of Γ in D_u; then $(Z_u)_{u \geqslant 0}$ is one parametrized form of Γ, as Walsh [17] defined it. (Sometimes called canonical parametrization.)

II. The Past of a Stopping Point

Without any doubt $[\![Z, \infty[\![$ is the future of a stopping point Z, but $[\![0, Z]\!]$ cannot be, in general, the past of Z. ($I_{[\![0,Z]\!]}$ may not be adapted!) In this section we propose a definition of the past of a stopping point which seems to be the natural extension of the one-parameter case.

The next result gives a sense to this definition (these results were announced in [10]):

Proposition 4. *Let Γ be an optional increasing path, passing through Z: then we have $\vec{\Gamma} \subset \vec{\Gamma}_Z$ and $\bar{\Gamma} \subset \bar{\gamma}_Z$; in particular $\vec{\gamma}_Z \subset \vec{\Gamma}_Z$.*

Proof. Given $z \in \mathbb{R}_+^2$, we define $A = \{\omega \in \Omega / z \in [(\vec{\Gamma}_Z \cup \vec{\Gamma}) \cap \vec{\Gamma}_Z^c](\omega)\}$; we have: $A \in \mathscr{F}_z$ and $I_A \cdot X_z^{+-} = E(I_{A \cap \{T < t\} \cap \{S > s\}} / \mathscr{F}_s^1)$; $A \cap \{T < t\} \cap \{S > s\}$ is negligible since $(\vec{\Gamma}_Z \cup \vec{\Gamma}) \cap R_{Z-+}^2$ is evanescent. On the other hand, outside $\vec{\Gamma}_Z, X^{+-} \neq 0$, therefore $P(A) = 0$. Since $I_{(\vec{\Gamma}_Z \cup \vec{\Gamma}) \cap \vec{\Gamma}_Z^c}$ is 4-continuous, $\vec{\Gamma}_Z \cup \vec{\Gamma}) \cap \vec{\Gamma}_Z^c$ is evanescent. An illustration of the situation is shown in Fig. 1.

Definition 5. The *past of a stopping point* Z is the optional set $\mathscr{P}_Z = (\vec{\Gamma}_Z \cap \bar{\gamma}_Z) \cap ([\![Z, \infty[\![^c \cup [\![Z]\!])$.

Proposition 6. $Z_1 \leqslant Z_2 \Leftrightarrow \mathscr{P}_{Z_1} \subset \mathscr{P}_{Z_2}$.

Proof. We suppose $\mathscr{P}_{Z_1} \subset \mathscr{P}_{Z_2}$: then $[\![Z_1]\!] \subset \mathscr{P}_{Z_1} \subset \mathscr{P}_{Z_2} \subset [\![0, Z_2]\!]$ implies $Z_1 \leqslant Z_2$ on $\{Z_1 < \infty\}$. On the other hand, $\{Z = \infty\} = \{\mathscr{P}_Z \text{ is unbounded}\}$, therefore on $\{Z_1 = \infty\}$, $\mathscr{P}_{Z_1}$ is unbounded, so is $\mathscr{P}_{Z_2}$ and $Z_2 = \infty$. Now we suppose $Z_1 \leqslant Z_2$; we have

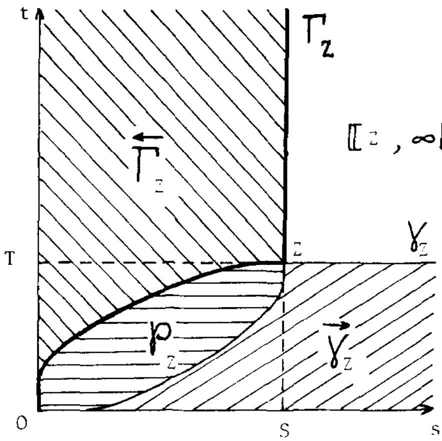

FIG. 1. The past of a stopping point.

$$\{X^{1+-}=0\} \cap ([\![Z_1, \infty[\![^c \cup [\![Z_1]\!]) \subset \{X^{2+-}=0\}$$
$$\{Y^{1-+}=0\} \cap ([\![Z_1, \infty[\![^c \cup [\![Z_1]\!]) \subset \{Y^{2-+}=0\}$$
$$([\![Z_1, \infty[\![^c \cup [\![Z_1]\!]) \subset ([\![Z_2, \infty[\![^c \cup [\![Z_2]\!]$$

and then $\mathscr{P}_{Z_1} \subset \mathscr{P}_{Z_2}$.

Remarks. (1) If $Z_1 \leqslant Z_2$, an easy consequence of Theorem 3 is: there exists an OIP Γ passing through Z_1 and Z_2. We have the same conclusion if Z_1 and Z_2 are only comparable (using $Z_1 \wedge Z_2$ and $Z_1 \wedge Z_2$).

(2) $\mathscr{P}_z = \Omega \times [0, z]$

(3) A stopping point Z defines four "quadrants":

$$\tilde{R}^1_Z = R^1_Z = [\![Z, \infty[\![$$
$$\tilde{R}^3_Z = \mathscr{P}_Z \qquad (\subset R^3_Z)$$
$$\tilde{R}^2_Z = \{X^{+-} > 0\} \qquad \text{and} \qquad \tilde{R}^4_Z = \{Y^{-+} > 0\}.$$

These four sets are optional; we can express the set where another stopping point Z_1 belongs to one of these quadrants:

$$\{Z_1 \in \tilde{R}^1_Z\} = \{Z \leqslant Z_1\}, \qquad \{Z_1 \in \mathscr{P}_Z\}$$
$$\{Z_1 \in \tilde{R}^2_Z\} = \{X^{+-}_{Z_1} > 0\} \qquad \text{and} \qquad \{Z_1 \in \tilde{R}^4_Z\} = \{Y^{-+}_{Z_1} > 0\};$$

these four sets are $\mathscr{F}_{Z_1}$-measurable. It will be useful to notice that $X^{+-}_{Z_1} = I_{\{S > S_1\}} E(I_{\{T < T_1\}} / \mathscr{F}^1_{S_1})$ and $Y^{-+}_{Z_1} = I_{\{T > T_1\}} E(I_{\{S < S_1\}} / \mathscr{F}^2_{T_1})$. To show that for the first one, for instance, we use the proof of Theorem 3; we know it for Z_1, a simple 1-stopping point; by the same computation, this is true for Z_1, a discrete 1-stopping point (∞ is a possible value). Any stopping point Z_1 is recalled in R^4 by a sequence of discrete 1-stopping points; we conclude by the 4-continuity.

III. Infimum of Two Stopping Points and Associated σ-Algebra

In [10], we defined the infimum of a stopping point and a point; here is the generalization to two stopping points:

Definition 7. *Let* Z_1 and Z_2 be two stopping points. The entrance point in $(\mathscr{P}_{Z_1} \cap \mathscr{P}_{Z_2})^c$ of the OIP associated with $\vec{\Gamma}_{Z_1} \cap \vec{\Gamma}_{Z_2}$ is equalled to the entrance point in $(\mathscr{P}_{Z_1} \cap \mathscr{P}_{Z_2})^c$ of the OIP associated with $\bar{\gamma}_{Z_1} \cap \bar{\gamma}_{Z_2}$; we call it *the infimum of* Z_1 *and* Z_2 and we denote it by: $Z_1 \dot{\wedge} Z_2$ (we have to make a distinction with the geometrical infimum $Z_1 \wedge Z_2$ which is not, in general, a stopping point).

Remark. $Z_1 \leqslant Z_2$ implies $Z_1 \dot{\wedge} Z_2 = Z_1$. The next result is the justification of this definition:

Proposition 8. *Let* Z_1 *and* Z_2 *be two stopping points; then we have* $\mathscr{P}_{Z_1 \dot{\wedge} Z_2} = \mathscr{P}_{Z_1} \cap \mathscr{P}_{Z_2}$ *and* $Z_1 \dot{\wedge} Z_2$ *is the largest stopping point smaller than* Z_1 *and* Z_2.

Proof. The OIP associated with $\vec{\Gamma}_{Z_1} \cap \vec{\Gamma}_{Z_2}$ passes through $Z_1 \dot{\wedge} Z_2$; the OIP associated with $\bar{\gamma}_{Z_1} \cap \bar{\gamma}_{Z_2}$ passes through $Z_1 \dot{\wedge} Z_2$. Therefore $\mathscr{P}_{Z_1} \cap \mathscr{P}_{Z_2} \subset \mathscr{P}_{Z_1 \dot{\wedge} Z_2}$; but $Z_1 \dot{\wedge} Z_2 \leqslant Z_1$, $Z_1 \dot{\wedge} Z_2 \leqslant Z_2$ and Proposition 6 show that $\mathscr{P}_{Z_1 \dot{\wedge} Z_2} \subset \mathscr{P}_{Z_1} \cap \mathscr{P}_{Z_2}$. Let Z_3 be another stopping point. We have $(Z_1 \dot{\wedge} Z_2) \dot{\wedge} Z_3 = Z_1 \dot{\wedge} (Z_2 \dot{\wedge} Z_3)$; $(Z_1 \dot{\wedge} Z_2) \dot{\wedge} Z_3$ is, in fact, the entrance point in $(\mathscr{P}_{Z_1 \dot{\wedge} Z_2} \cap \mathscr{P}_{Z_3})^c = (\mathscr{P}_{Z_1} \cap \mathscr{P}_{Z_2} \cap \mathscr{P}_{Z_3})^c$ of the OIP associated with $\vec{\Gamma}_{Z_1} \cap \vec{\Gamma}_{Z_2} \cap \vec{\Gamma}_{Z_3}$ or $\bar{\gamma}_{Z_1} \cap \bar{\gamma}_{Z_2} \cap \bar{\gamma}_{Z_3}$. Let Z be a stopping point such that $Z \leqslant Z_1$ and $Z \leqslant Z_2$; then we have $Z = Z \dot{\wedge} Z = (Z \dot{\wedge} Z_1) \dot{\wedge} (Z \dot{\wedge} Z_2) = Z \dot{\wedge} (Z_1 \dot{\wedge} Z_2)$ and therefore $Z \leqslant Z_1 \dot{\wedge} Z_2$.

Remarks. (1) $z \dot{\wedge} z' = z \wedge z'$.

(2) Z_1 and Z_2 comparable implies $Z_1 \dot{\wedge} Z_2 = Z_1 \wedge Z_2$; more generally, if $Z_1 \wedge Z_2$ is a stopping point, then $Z_1 \dot{\wedge} Z_2 = Z_1 \wedge Z_2$.

(3) The complete study of two examples may help:

(a) Let z_0 be in $\mathbb{R}_+^2$ and $A \in \mathscr{F}_{z_0}$; we set: $Z = z_0^A = z_0$ on A and ∞ on A^c. This example will show that on $\{Z = \infty\}$, $Z \dot{\wedge} z$ may not be z, but $z \wedge z_0$.

(b) The example given in [17]: $(\mathscr{F}_z)$ is generated by the brownian sheet (W_z); $Z=(1,2)$ on $\{W_{11}\geqslant 0\}$ and $(2,1)$ on $\{W_{11}<0\}$. On $\{W_{11}\geqslant 0\}$, $\mathscr{P}_Z(\omega)=[0,(1,1)]\cup[(1,1),(1,2)]$ and on $\{W_{11}<0\}$, $\mathscr{P}_Z(\omega)=[0,(1,1)]\cup[(1,1),(2,1)]$. Then, for instance $(0,2)\dot{\wedge} Z=(0,1)$.

Without using any martingale theory we characterize the σ-algebra $\mathscr{F}_{Z\dot{\wedge} z}$, where Z is a stopping point and z a point in $\mathbb{R}^2_+$.

PROPOSITION 9. $\mathscr{F}_{Z\dot{\wedge} z}=\mathscr{F}_Z\cap\mathscr{F}_z$.

Proof. $Z\dot{\wedge} z$ is a stopping point smaller than Z and z; therefore $\mathscr{F}_{Z\dot{\wedge} z}\subset\mathscr{F}_Z\cap\mathscr{F}_z$. Let A be a set in $\mathscr{F}_Z\cap\mathscr{F}_z$; we study $B=A\cap\{Z\dot{\wedge} z\leqslant(u,v)\}$ where $(u,v)\in\mathbb{R}^2_+$. Using the four "quadrants" defined by Z, we have: $B=[B\cap(\tilde{R}^1_Z\cap\{Z\neq z\})\cup(B\cap\tilde{R}^2_Z)\cup(B\cap\tilde{R}^3_Z)\cup(B\cap\tilde{R}^4_Z)$.

(1) $B\cap(\tilde{R}^1_Z\cap\{Z\neq z\})=A\cap\{Z\dot{\wedge} z\leqslant(u,v)\}\cap\{Z\leqslant z\}\cap\{Z\neq z\}=A\cap\{Z\leqslant(u,v)\}\cap\{Z\leqslant z\}\cap\{Z\neq z\}$ (because on $\{Z\leqslant z\}$, $Z\dot{\wedge} z=Z$) $=(A\cap\{Z\neq z\})\cap\{Z\leqslant(u,v)\}\cap\{Z\leqslant z\}$; $A\cap\{Z\neq z\}\in\mathscr{F}_Z$ and $\{Z\leqslant(u,v)\}\cap\{Z\leqslant z\}=\{Z\leqslant(u,v)\wedge z\}$ imply $B\cap(\tilde{R}^1_Z\cap\{Z\neq z\})\in\mathscr{F}_{(u,v)\wedge z}\subset\mathscr{F}_{u,v}$.

(2) $B\cap\tilde{R}^2_Z=A\cap\{Z\dot{\wedge} z\leqslant(u,v)\}\cap\{X^{+-}_z>0\}=A\cap\{s\leqslant u\}\cap\{X^{+-}_{s,t\wedge v}>0\}\cap\{X^{+-}_z>0\}$ (because on $\{X^{+-}_z>0\}$ $S_{Z\dot{\wedge} z}=s$ and $\{Z\dot{\wedge} z\leqslant(u,v)\}\cap\{X^{+-}_z>0\}=\{s\leqslant u\}\cap\{T_{Z\dot{\wedge} z}\leqslant v\}\cap\{X^{+-}_z>0\}$ but on $\{X^{+-}_z>0\}$, $T_{Z\dot{\wedge} z}\leqslant t$). Then $B\cap\tilde{R}^2_Z=A\cap\{s\leqslant u\}\cap\{X^{+-}_{s,t\wedge v}>0\}$ (because $(X^{+-}_{s,t})_{t\geqslant 0}$ is increasing). $B\cap\tilde{R}^2_Z=\{I_{A\cap\{S>s\}}E(I_{\{T<t\wedge v\}}/\mathscr{F}^1_s)>0\}\cap\{s\leqslant u\}$ $=\{E(I_{A\cap\{S>s\}\cap\{T<t\wedge v\}}/\mathscr{F}^1_s)>0\}\cap\{s\leqslant u\}$ (because $A\in\mathscr{F}_z\subset\mathscr{F}^1_s$ and $\{S>s\}\in\mathscr{F}^1_s$) $=\{E(I_{(A\cap\{T<t\wedge v\})\cap(\{S>s\}\cap\{T<t\wedge v\})}/\mathscr{F}^1_s)>0\}\cap\{s\geqslant u\}$; $A\in\mathscr{F}_Z\subset\mathscr{F}^2_T$ implies $A\cap\{T<t\wedge v\}\in\mathscr{F}^2_{t\wedge v}$ and $\{S>s\}\cap\{T<t\wedge v\}=\{Z\in R^4_{(s,t\wedge v)+-}\}\in\mathscr{F}^2_{t\wedge v}$, then using F_4: $B\cap\tilde{R}^2_Z=\{E(I_{A\cap\{T<t\wedge v\}\cap\{S>s\}}/\mathscr{F}_{s,t\wedge v})>0\}\cap\{s\leqslant u\}\in\mathscr{F}_{u,t\wedge v}\subset\mathscr{F}_{u,v}$.

(3) $B\cap\tilde{R}^3_Z=B\cap\{z\in\mathscr{P}_Z\}=A\cap\{Z\dot{\wedge} z\leqslant(u,v)\}\cap\{z\in\mathscr{P}_Z\}=A\cap\{z\leqslant(u,v)\}\cap\{z\in\mathscr{P}_Z\}$ (because on $\{z\in\mathscr{P}_Z\}$, $Z\dot{\wedge} z=z$) $=(A\cap\{z\in\mathscr{P}_Z\})\cap\{z\leqslant(u,v)\}$; $A\in\mathscr{F}_z$, $\{z\in\mathscr{P}_Z\}\in\mathscr{F}_z$ and $z\leqslant(u,v)$ imply $B\cap\tilde{R}^3_Z\in\mathscr{F}_{u,v}$.

(4) Is symmetrical. Therefore $A\cap\{Z\dot{\wedge} z\leqslant(u,v)\}\in\mathscr{F}_{u,v}$ for every $(u,v)\in\mathbb{R}^2_+$ and then $A\in\mathscr{F}_{Z\dot{\wedge} z}$; so $\mathscr{F}_Z\cap\mathscr{F}_z\subset\mathscr{F}_{Z\dot{\wedge} z}$. In order to study the properties of these σ-algebras we need to stop martingales at a stopping point Z.

IV. STOPPING FOR MARTINGALES

DEFINITION 10. Let (M_z) a process and Z a stopping point: we call $M^Z=(M_{Z\dot{\wedge} z},\ z\in\mathbb{R}^2_+)$ the *process M stopped at Z*.

Remark. $Z\dot{\wedge} z$ is a bounded stopping point and then M^Z is well defined.

If M is progressive, M^Z is adapted to $(\mathscr{F}_{Z \dot{\wedge} z})_{z \in \mathbb{R}_+^2}$. First, we study a stopped martingale at Z with respect to this filtration.

PROPOSITION 11. *Let (M_z) be a right continuous martingale and Z a stopping point; then M^Z is a right continuous martingale with respect to the filtration $(\mathscr{F}_z^Z = \mathscr{F}_{Z \dot{\wedge} z} = \mathscr{F}_Z \cap \mathscr{F}_z)_{z \in \mathbb{R}_+^2}$.*

Proof. Given $z \leqslant z'$, we have: $E(M_{z'}^Z / \mathscr{F}_z^Z) = E(M_{Z \dot{\wedge} z'} / \mathscr{F}_{Z \dot{\wedge} z}) = M_{Z \dot{\wedge} z} = M_z^Z$ a.s., because $Z \dot{\wedge} z$ and $Z \dot{\wedge} z'$ are bounded stopping points such that $Z \dot{\wedge} z \leqslant Z \dot{\wedge} z'$ and the optional sampling theorem for right continuous martingales.

The next result characterizes the σ-algebra $\mathscr{F}_{Z_1}^Z$ where Z_1 is an $(\mathscr{F}_z^Z)$-stopping point.

PROPOSITION 12. *Let Z_1 be an $(\mathscr{F}_z^Z)$-stopping point: then $\mathscr{F}_{Z_1}^Z = \mathscr{F}_{Z \dot{\wedge} Z_1} = \mathscr{F}_Z \cap \mathscr{F}_{Z_1}$.*

Proof. An $(\mathscr{F}_z^Z)$-stopping point is clearly an $\mathscr{F}_Z$-measurable stopping point. We suppose first, that Z_1 is discrete with Range $(Z) = \{z_n\}$; we denote by $Z \mathbin{\overset{\vdots}{\wedge}} Z_1$ the random point defined by $(Z \mathbin{\overset{\vdots}{\wedge}} Z_1)(\omega) = (Z \dot{\wedge} Z_1(\omega))(\omega)$. $\{Z \dot{\wedge} Z_1 \leqslant z\} = \bigcup_n (\{Z \dot{\wedge} Z_1 \leqslant z\} \cap \{Z_1 = z_n\}) = \bigcup_n (\{Z \dot{\wedge} z_n \leqslant z\} \cap \{Z_1 = z_n\})$; $\{Z_1 = z_n\} \in \mathscr{F}_{Z \dot{\wedge} z_n}$ and then $\{Z \dot{\wedge} z_n \leqslant z\} \cap \{Z_1 = z_n\} \in \mathscr{F}_z$. Therefore $Z \mathbin{\overset{\vdots}{\wedge}} Z_1$ is a stopping point; by construction we have $Z \dot{\wedge} Z_1 \leqslant Z \mathbin{\overset{\vdots}{\wedge}} Z_1$, $Z \mathbin{\overset{\vdots}{\wedge}} Z_1 \leqslant Z$ and $Z \mathbin{\overset{\vdots}{\wedge}} Z_1 \leqslant Z_1$; Proposition 8 gives us $Z \dot{\wedge} Z_1 = Z \mathbin{\overset{\vdots}{\wedge}} Z_1$.

Any $(\mathscr{F}_z^Z)$-stopping point Z_1 is recalled in R^1 by a sequence (Z_1^n) of discrete $(\mathscr{F}_z^Z)$-stopping points and then $Z \dot{\wedge} Z_1^n = Z \mathbin{\overset{\vdots}{\wedge}} Z_1^n$ implies $Z \dot{\wedge} Z_1 = Z \mathbin{\overset{\vdots}{\wedge}} Z_1$ (we have to notice that $\mathscr{P}_{Z_1^n} \downarrow \mathscr{P}_{Z_1}$). $\mathscr{F}_Z \cap \mathscr{F}_{Z_1} = \{A \in \mathscr{F}_Z / A \in \mathscr{F}_{Z_1}\} = \{A \in \mathscr{F}_Z / A \cap \{Z_1 \leqslant z\} \in \mathscr{F}_z, \forall z \in \mathbb{R}_+^2\} = \{A \in \mathscr{F}_Z / A \cap \{Z_1 \leqslant z\} \in \mathscr{F}_Z \cap \mathscr{F}_z = \mathscr{F}_z^Z, \forall z \in \mathscr{R}_+^2\}$ (the last equality because if $A \in \mathscr{F}_Z$, $A \cap \{Z_1 \leqslant z\} \in \mathscr{F}_Z$). $\mathscr{F}_\infty^Z = \mathscr{F}_Z$ shows that $\mathscr{F}_{Z_1}^Z = \mathscr{F}_Z \cap \mathscr{F}_{Z_1}$. Since $Z \dot{\wedge} Z_1$ is a stopping point smaller than Z and Z_1, we have $\mathscr{F}_{Z \dot{\wedge} Z_1} \subset \mathscr{F}_Z \cap \mathscr{F}_{Z_1}$.

Let A be in $\mathscr{F}_Z \cap \mathscr{F}_{Z_1}$. We set $M_z = E(I_A / \mathscr{F}_z)$ (right continuous version); we know, by Proposition 11, that M^Z is an $(\mathscr{F}_z^Z)$-martingale. We have $M_z^Z = M_{Z \dot{\wedge} z} = E(I_A / \mathscr{F}_{Z \dot{\wedge} z}) = E(I_A / \mathscr{F}_z^Z)$; then $M_{Z_1}^Z = E(I_A / \mathscr{F}_{Z_1}^Z) = I_A$ (since $\mathscr{F}_{Z_1}^Z = \mathscr{F}_Z \cap \mathscr{F}_{Z_1}$ and $A \in \mathscr{F}_Z \cap \mathscr{F}_{Z_1}$).

On the other hand, $M_{Z_1}^Z(\omega) = M_{Z \dot{\wedge} Z_1(\omega)}(\omega) = M_{Z \mathbin{\overset{\vdots}{\wedge}} Z_1}(\omega) = M_{Z \dot{\wedge} Z_1}(\omega)$ and therefore $M_{Z \dot{\wedge} Z_1} = I_A$ which implies $A \in \mathscr{F}_{Z \dot{\wedge} Z_1}$. The next result shows that the conditional independence property is preserved by stopping at a stopping point Z.

PROPOSITION 13. *Let Z be a stopping point: the filtration* $(\mathscr{F}_z^Z = \mathscr{F}_{Z \dot{\wedge} z} = \mathscr{F}_Z \cap \mathscr{F}_z)_{z \in \mathbb{R}_+^2}$ verifies F_4.

Proof. Let u and v be positive numbers; we introduce two stopping points: $Z_1 = Z \dot{\wedge} (s, t+v)$ and $Z_2 = Z \dot{\wedge} (s+u, t)$. Figures 2–4 help illustrate how Z_1 and Z_2 are constructed; they are classified in 3 cases according to the remaining of the proof. This classification is obviously related to the following partition of Ω: $(\{z \in \mathscr{P}_Z\}, \{Z_1 \in \mathscr{P}_{Z_2}\} \cap \{Z_1 \neq z\}, \{Z_2 \in \mathscr{P}_{Z_1}\} \cap \{Z_2 \neq Z_1\} \cap \{Z_2 \neq z\})$. We have $(\mathscr{F}_z^Z)^1 = V_{v>0} \mathscr{F}_{s,t+v}^Z = V_{v>0} (\mathscr{F}_Z \cap \mathscr{F}_{s,t+v}) = V_{v>0} \mathscr{F}_{Z \dot{\wedge} (s,t+v)} = V_{v>0} \mathscr{F}_{Z_1}$, where Z_1 depends clearly in v. For the same reason,

$$(\mathscr{F}_z^Z)^2 = \bigvee_{u>0} \mathscr{F}_{Z \dot{\wedge} (s+u,t)} = \bigvee_{u>0} \mathscr{F}_{Z_2}.$$

We will show that for every $z \in \mathbb{R}_+^2$ and u, v positive numbers, $\mathscr{F}_{Z_1}$ and $\mathscr{F}_{Z_2}$ are conditionally independent given $\mathscr{F}_z^Z = \mathscr{F}_{Z \dot{\wedge} z}$. Let Y be a bounded, $\mathscr{F}_{Z_2}$-measurable random variable: we set $M_z = E(Y/\mathscr{F}_z)$ (the right continuous version); then $M_{Z_2} = E(Y/\mathscr{F}_{Z_2}) = Y$ and $E(Y/\mathscr{F}_{Z \dot{\wedge} z}) = E(M_{Z_2}/\mathscr{F}_{Z \dot{\wedge} z}) = M_{Z \dot{\wedge} z}$ (we have $Z \dot{\wedge} z \leqslant Z_2$); therefore $E(Y/\mathscr{F}_{Z_1}) = E(Y/\mathscr{F}_{Z \dot{\wedge} z})$ is equivalent to $E(M_{Z_2}/\mathscr{F}_{Z_1}) = M_{Z \dot{\wedge} z}$. Given any $A \in \mathscr{F}_{Z_1}$ we will show that $E(M_{Z_2} : A) = E(M_{Z \dot{\wedge} z} : A)$. We decompose this equality according to the partition mentioned before:

(1) $E(M_{Z_2} : A \cap \{z \in \mathscr{P}_Z\}) = E(M_{Z_2 \vee z} : A \cap \{z \in \mathscr{P}_Z\})$ (on $\{z \in \mathscr{P}_Z\}$, $Z_2 = Z_2 \vee z$) $A \in \mathscr{F}_{Z_1}$, $\{z \in \mathscr{P}_Z\} \in \mathscr{F}_Z \cap \mathscr{F}_z = \mathscr{F}_{Z \dot{\wedge} z} \subset \mathscr{F}_{Z_1}$ imply $A \cap \{z \in \mathscr{P}_Z\} \in \mathscr{F}_{Z_1} \subset \mathscr{F}_{Z_1 \vee z}$ $(Z_1 \vee z) \dot{\wedge} (Z_2 \vee z) = (Z_1 \vee z) \wedge (Z_2 \vee z) = z$ and we have the commutation $E(\cdot/\mathscr{F}_{Z_1 \vee z}/\mathscr{F}_{Z_2 \vee z}) = E(\cdot/\mathscr{F}_{Z_2 \vee z}/\mathscr{F}_{Z_1 \vee z}) = E(\cdot/\mathscr{F}_z)$,

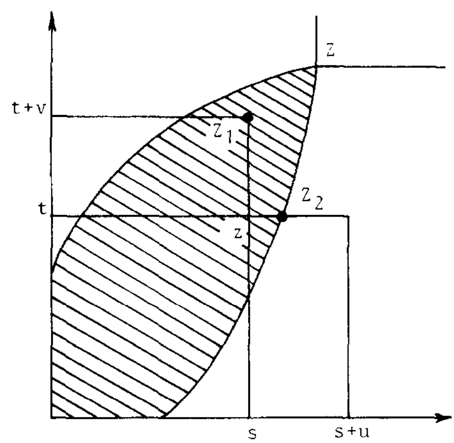

FIG. 2. $\{z \in \mathscr{P}_Z\}$.

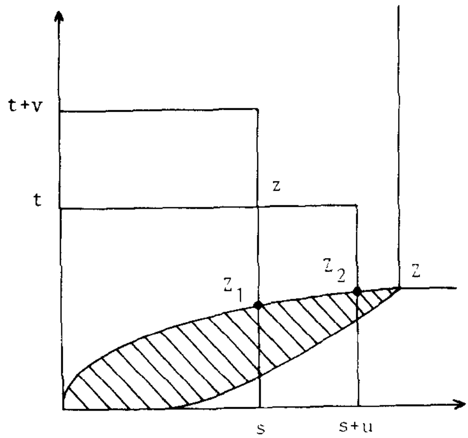

FIG. 3. $\{Z_1 \in \mathscr{P}_{Z_2}\} \cap \{Z_1 \neq z\}$.

which shows that $E(M_{Z_2 \vee z}/\mathscr{F}_{Z_1 \vee z}) = M_z$. Therefore $E(M_{Z_2}: A \cap \{z \in \mathscr{P}_Z\}) = E(M_z: A \cap \{z \in \mathscr{P}_Z\}) = E(M_{Z \dot{\wedge} z}: A \cap \{z \in \mathscr{P}_Z\})$ (on $\{z \in \mathscr{P}_Z\}$, $z = Z \wedge z$).

(2) $E(M_{Z_2}: A \cap \{Z_1 \in \mathscr{P}_{Z_2}\} \cap \{Z_1 \neq z\})$ $A \in \mathscr{F}_{Z_1}$, $\{Z_1 \neq z\} \in \mathscr{F}_{Z_1}$, so $A \cap \{Z_1 \neq z\} \in \mathscr{F}_{Z_1}$; we define

$$Z_3 = \begin{matrix} Z_2 & \text{on} & \{Z_1 \in \mathscr{P}_{Z_2}\} \\ \infty & \text{on} & \{Z_1 \in \mathscr{P}_{Z_2}\}^c \end{matrix};$$

Z_3 is a stopping point and $Z_1 \leqslant Z_3$ then $E(M_{Z_2}: A \cap \{Z_1 \neq z\} \cap \{Z_1 \in \mathscr{P}_{Z_2}\})$

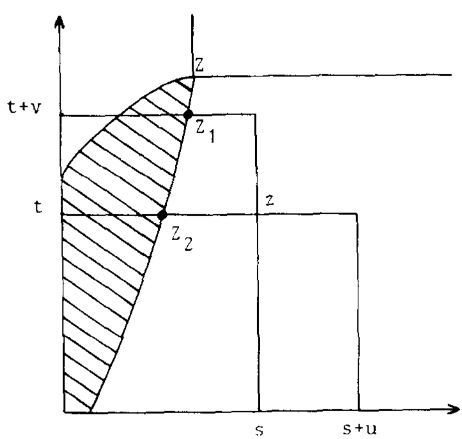

FIG. 4. $\{Z_2 \in \mathscr{P}_{Z_1}\} \cap \{Z_1 \neq Z_2\} \cap \{Z_2 \neq z\}$.

$= E(M_{Z_3}\colon A \cap \{Z_1 \neq z\} \cap \{Z_1 \in \mathscr{P}_{Z_2}\}) = E(M_{Z_1}\colon A \cap \{Z_1 \neq z\} \cap \{Z_1 \in \mathscr{P}_{Z_2}\})$ (since $\{Z_1 \in \mathscr{P}_{Z_2}\} \in \mathscr{F}_{Z_1}$ too). $= E(M_{Z \dot{\wedge} z}\colon A \cap \{Z_1 \neq z\} \cap \{Z_1 \in \mathscr{P}_{Z_2}\})$ (on $\{Z_1 \in \mathscr{P}_{Z_2}\}$, $Z_1 = Z \dot{\wedge} z$)

(3) $E(M_{Z_2}\colon A \cap \{Z_2 \in \mathscr{P}_{Z_1}\} \cap \{Z_2 \neq Z_1\} \cap \{Z_2 \neq z\}) = E(M_{Z \dot{\wedge} z}\colon A \cap \{Z_2 \in \mathscr{P}_{Z_1}\} \cap \{Z_2 \neq Z_1\} \cap \{Z_2 \neq z\})$ (because on $\{Z_2 \in \mathscr{P}_{Z_1}\}$, we have: $Z_2 = Z \dot{\wedge} z$).

Proposition 11 tells us that a right continuous martingale M, stopped at a stopping point Z is a martingale with respect to the stopped filtration $(\mathscr{F}_z^Z)$. What can we say about (M_z^Z) with respect to the initial filtration $(\mathscr{F}_z)$? One must be careful: in general M^Z is not a martingale.

Example (b) given in Section III illustrates this fact: $(\mathscr{F}_z)$ is generated by the brownian sheet (W_z); we set $\{W_{1,1} < 0\} = A$ and $Z = (1,2) I_{A^c} + (2,1) I_A$; for any (s,t) such that $0 \leqslant s < 1$ and $t \geqslant 1$ we have $(s,t) \dot{\wedge} Z = (s,1)$. If (W_z^Z) is a martingale, applying the martingale property between (s,t) and $(2,2)$ with $t \leqslant 2$, we have:

$$\begin{aligned} W_{s,1} = W_{(s,t) \dot{\wedge} Z} = W^Z_{s,t} &= E(W^Z_{2,2}/\mathscr{F}_{s,t}) = E(W_Z/\mathscr{F}_{s,t}) \\ &= E(W_{2,1} I_A/\mathscr{F}_{s,t}) + E(W_{1,2} I_{A^c}/\mathscr{F}_{s,t}). \end{aligned}$$

Using F_4 and introducing $\mathscr{F}_{1,1}$ in the first term:

$$\begin{aligned} W_{s,1} &= E(W_{2,1} I_A/\mathscr{F}_{s,1}) + E(W_{1,2} I_{A^c}/\mathscr{F}_{s,t}) \\ &= E(E(W_{2,1}/\mathscr{F}_{1,1}) I_A/\mathscr{F}_{s,1}) + E(W_{1,2} I_{A^c}/\mathscr{F}_{s,t}) \\ &= E(W_{1,1} I_A/\mathscr{F}_{s,1}) + E(W_{1,2} I_{A^c}/\mathscr{F}_{s,t}). \end{aligned}$$

In this filtration the $L \log L$-bounded martingales being continous (here in the rectangle $[(0,0),(2,2)]$), let s go to 1 and t to 2:

$$\begin{aligned} W_{1,1} &= E(W_{1,1} I_A/\mathscr{F}_{1,1}) + E(W_{1,2} I_{A^c}/\mathscr{F}_{1,2}) \\ &= W_{1,1} I_A + W_{1,2} I_{A^c} \end{aligned}$$

which implies $W_{1,1} = W_{1,2}$ on A^c which is false. At least, M^Z is a weak martingale:

THEOREM 14. *Let M be a right continuous martingale and Z a stopping point; then M^Z, the stopped process, is a* weak martingale (*with respect to the initial filtration $(\mathscr{F}_z)$*).

Proof. $z \in \mathbb{R}_+^2$ and u, v are two positive numbers; we set $z' = (s+u, t+v)$. $M^Z]z, z'] = M_{Z \dot{\wedge} z'} - M_{Z \dot{\wedge} (s,t+v)} - M_{Z \dot{\wedge} (s+u,t)} + M_{Z \dot{\wedge} z} = M_{Z \dot{\wedge} z'} - M_{Z_1} - M_{Z_2} + M_{Z \dot{\wedge} z}$, where Z_1 and Z_2 are as in the proof of Proposition 13. $(\{Z \leqslant z\} \cap \{Z \neq z\}, \{X_z^{+-} > 0\}, \{z \in \mathscr{P}_Z\}, \{Y_z^{-+} > 0\})$ is a partition of Ω (the four "quadrants" associated with Z): we denote it (A_1, A_2, A_3, A_4), each A_i being $\mathscr{F}_z$-measurable.

(1) $E(M^Z]z,z']/\mathscr{F}_z)I_{A_1} = E[(M_{Z\dot\wedge z'} - M_{Z_1} - M_{Z_2} + M_{Z\dot\wedge z})I_{A_1}/\mathscr{F}_z]$
$= 0$ because on A_1, $Z \dot\wedge z' = Z_1 = Z_2 = Z \dot\wedge z = Z$

(2) $E(M^Z]z,z']/\mathscr{F}_z)I_{A_2} = E[(M_{Z\dot\wedge z'} - M_{Z_1} - M_{Z_2} + M_{Z\dot\wedge z})I_{A_2}/\mathscr{F}_z]$
$= E[(M_{Z\dot\wedge z'} - M_{Z_2})I_{A_2}/\mathscr{F}_z]$ (on A_2, $Z \dot\wedge z = Z_1$)
$= E[(M_{Z\dot\wedge z'} - M_{Z_2})I_{A_2}/\mathscr{F}_{Z_2\vee z}/\mathscr{F}_z]$
$= E[(M_{Z\dot\wedge z'} - M_{Z_2})I_{A_2}I_{\{Z_2 = Z_2\vee z\}}/\mathscr{F}_{Z_2\vee z}/\mathscr{F}_z]$
$+ E[(M_{Z\dot\wedge z'} - M_{Z_2})I_{A_2}I_{\{Z_2 \neq Z_2\vee z\}}/\mathscr{F}_{Z_2\vee z}/\mathscr{F}_z]$
$= E[(M_{(Z\dot\wedge z')\vee(Z_2\vee z)} - M_{Z_2\vee z})I_{A_2}I_{\{Z_2 = Z_2\vee z\}}/\mathscr{F}_{Z_2\vee z}/\mathscr{F}_z]$

because on $A_2 \cap \{Z_2 = Z_2 \vee z\}$ we have $Z_2 = Z_2 \vee z$ and $Z_2 \vee z \leqslant Z \dot\wedge z \leqslant Z \dot\wedge z'$; on $A_2 \cap \{Z_2 \neq Z_2 \vee z\}$ we have $Z \dot\wedge z' = Z_2$. Then $E(M^Z]z,z']/\mathscr{F}_z)I_{A_2} = E\{[E(M_{(Z\dot\wedge z')\vee(Z_2\vee z)} - M_{Z_2\vee z}/\mathscr{F}_{Z_2\vee z})I_{A_2} I_{\{Z_2 = Z_2\vee z\}}]/\mathscr{F}_z\} = 0$ by the martingale property applied between $Z_2 \vee z \leqslant (Z \dot\wedge z') \vee (Z_2 \vee z)$, bounded stopping points.

(3) $E(M^Z]z,z']/\mathscr{F}_z)I_{A_3} = E[(M_{Z\dot\wedge z'} - M_{Z_1} - M_{Z_2} + M_{Z\dot\wedge z})I_{A_3}/\mathscr{F}_z] = E(M_{(Z\dot\wedge z')\vee(Z_2\vee z)} - M_{Z_2\vee z}/\mathscr{F}_{Z_2\vee z}/\mathscr{F}_z)I_{A_3} - E(M_{Z_1\vee z} - M_z/\mathscr{F}_z)I_{A_3}$ because on $A_3: Z_2 = Z_2 \vee z$, $Z_2 \vee z \leqslant Z \dot\wedge z'$ and $Z \dot\wedge z = z$, $z \leqslant Z_1$. Then $E(M^Z]z,z']/\mathscr{F}_z)I_{A_3} = 0$ by the martingale property applied between $Z_2 \vee z \leqslant (Z \dot\wedge z') \vee (Z_2 \vee z)$ and $z \leqslant Z_1 \vee z$, all bounded stopping points.

(4) as (2) $E(M^z]z,z']/\mathscr{F}_z)I_{A_4} = 0$ therefore $E(M^Z]z,z']/\mathscr{F}_z) = 0$.

Remark. We only need that M^Z is a right continuous martingale with respect to the stopped filtration $(\mathscr{F}_z^Z)$. In order to see that, we improve the previous proof using the following facts: $A_2 \cap \{Z_2 = Z_2 \vee z\} = A_2 \cap \{Z_2 \vee z \in \mathscr{P}_Z\}$ $A_3 \subset \{Z_2 \vee z = Z_2\}$, $A_3 = \{z \in \mathscr{P}_Z\}$ and in general: $E(M_Z/\mathscr{F}_z)I_{\{z\in\mathscr{P}_Z\}} = M_{Z\dot\wedge z}I_{\{z\in\mathscr{P}_Z\}}$. One may notice the difference with the one-parameter case: *one-parameter*:

$(M_t^T)_{t\geqslant 0}$ is a martingale with respect to $(\mathscr{F}_t^T)$ iff it is a martingale with respect to $(\mathscr{F}_t)$ (where T is an $(\mathscr{F}_t)$-stopping time and $E(\cdot/\mathscr{F}_T/\mathscr{F}_t) = E(\cdot/\mathscr{F}_t/\mathscr{F}_T) = E(\cdot/\mathscr{F}_{T\wedge t}) = E(\cdot/\mathscr{F}_t^T)$. *two-parameter*:

(M_z^Z) is a martingale with respect to $(\mathscr{F}_z^Z)$ implies that (M_z^Z) is a weak martingale with respect to $(\mathscr{F}_z)$.

V. Localization

As mentioned in the Introduction we do not intend to give the right definition of a local martingale (we do not know if a such definition exists!). The definition we give here is associated with the stopping method we described before. Proposition 16 is an encouraging result about the behavior of a such process along horizontal or vertical lines. An example is given at the end of this section.

DEFINITION 15. A "*local martingale*" is a right continuous, adapted process M such that there exists an increasing sequence $(Z_n)_{n\in\mathbb{N}}$ of stopping points such that $\mathscr{P}_{Z_n}$ increases to $\Omega\times\mathbb{R}^2_+$ and each M^{Z_n} is a uniformly integrable martingale with respect to the filtration $(\mathscr{F}^{Z_n}_z)$.

Remarks. By the remark following Theorem 14, each M^{Z_n} is a weak martingale with respect to the filtration $(\mathscr{F}_z)$. If we do not assume the right continuity and we replace uniformly integrable by $L\log L$ bounded, then Proposition 13 tells us that each M^{Z_n} has a right continuous version: therefore the process M itself is right continuous (limited in the other quadrants).

PROPOSITION 16. *Let $(M,(Z_n))$ be a "local martingale". Then for each t* [resp. s], $(M_{s,t},\mathscr{F}_{s,t},\ t\geqslant 0$ [*resp.* $s\geqslant 0$]) *is a local martingale.*

Proof. For each n we define a new OIP, passing through Z_n, by: γ'_{Z_n} is γ_{Z_n} up to Z_n and Γ_{Z_n} after Z_n. Let S'_n be the $(\mathscr{F}_{s,t})_{s\geqslant 0}$-stopping time defined by $S'_n=\inf\{s\geqslant 0/(s,t)\in\vec{\gamma}'_{Z_n}\}$. We set:

$$\begin{aligned} S_n &= S'_n \quad \text{on} \quad \{(0,t)\in\mathscr{P}_{Z_n}\} \\ &= 0 \quad \text{on} \quad \{(0,t)\in\mathscr{P}_{Z_n}\}^c \end{aligned}$$

S_n is a $(\mathscr{F}_{s,t})_{s\geqslant 0}$-stopping time $(\{(0,t)\in\mathscr{P}_{Z_n}\}\in\mathscr{F}_{0,t})$ and then (S_n,t) defined by (S_n,t) on $\{S_n<\infty\}$ and ∞ on $\{S_n=\infty\}$ is a stopping point. $S_n\uparrow+\infty$ because $\mathscr{P}_{Z_n}\uparrow\Omega\times\mathbb{R}^2_+$ and we have to check that $(M_{s\wedge S_n,t}\cdot I_{S_n>0)},\mathscr{F}_{s,t},\ s\geqslant 0)$ is a uniformly integrable martingale. We have $M_{s\wedge S_n,t}\cdot I_{\{S_n>0\}}=M^{Z_n}_{s,t}\cdot I_{\{S_n>0\}}$, because $\{S_n>0\}\subset\{(s\wedge S_n,t)\in\mathscr{P}_{Z_n}\}$ and $Z_n\wedge(s,t)=(s\wedge S_n,t)$ on $\{S_n>0\}$.

For $s\leqslant s'$:

$$\begin{aligned} &E(M_{s'\wedge S_n,t}\cdot I_{\{S_n>0\}}-M_{s\wedge S_n,t}\cdot I_{\{S_n>0\}}/\mathscr{F}_{s,t}) \\ &= E(M^{Z_n}_{s',t}-M^{Z_n}_{s,t})\,I_{\{S_n>0\}} \\ &= E(M^{Z_n}_{s',t}-M^{Z_n}_{s,t}/\mathscr{F}_{s,t})\,I_{\{(s,t)\in\mathscr{P}_{Z_n}\}\cap\{S_n>0\}} \\ &\quad + E(M^{Z_n}_{s',t}-M^{Z_n}_{s,t}/\mathscr{F}_{s,t})\,I_{\{(s,t)\in\mathscr{P}_{Z_n}\}^c\cap\{S_n>0\}} \\ &= E(M^{Z_n}_{s',t}-M^{Z_n}_{s,t}/\mathscr{F}^{Z_n}_{s,t})\,I_{\{(s,t)\in\mathscr{P}_{Z_n}\}\cap\{S_n>0\}} \\ &\quad + E(M^{Z_n}_{s',t}-M^{Z_n}_{s,t}/\mathscr{F}_{s,t})\,I_{\{(s,t)\in\mathscr{P}_{Z_n}\}^c\cap\{S_n>0\}}=0; \end{aligned}$$

the first term by the martingale property of M^{Z_n} with respect to $(\mathscr{F}_z^{Z_n})$; the second term because on $\{(s,t)\in \mathscr{P}_{Z_n}\}^c \cap \{S_n > 0\}$, $M^{Z_n}_{s',t} = M^{Z_n}_{s,t}$.

EXAMPLE. This example has already been used by Cairoli [4] to show that there exists nonevanescent predictable sets A such that $P\{\omega: (\omega, Z(\omega)) \in A\} = 0$ for every stopping point Z.

Let $(B_s^1)_{s\geqslant 0}$ and $(B_t^2)_{t\geqslant 0}$ be brownian motions defined on $(\Omega_1, \mathscr{F}_1, P_1)$ and $(\Omega_2, \mathscr{F}_2, P_2)$, taking values in $\mathbb{R}^3$. Let $(\mathscr{F}_s^1)_{s\geqslant 0}$ and $(\mathscr{F}_t^2)_{t\geqslant 0}$ be their natural filtrations: we set: $z=(s,t)$, $\omega=(\omega_1,\omega_2)$, $B_z=(B_s^1, B_t^2)$, $\Omega=\Omega_1\times\Omega_2$, $P=P_1\times P_2$ $\mathscr{F} \overset{P}{=} \mathscr{F}_1\times\mathscr{F}_2$ and $\mathscr{F}_z \overset{P}{=} \mathscr{F}_s^1\times\mathscr{F}_t^2$ ($\overset{P}{=}$ means that the P-negligeable sets are included). Let a be a point of $\mathbb{R}^3$ different from 0. We define: $f\colon \mathbb{R}^3\times\mathbb{R}^3 \to \overline{\mathbb{R}}_+$ by

$$f(x,y) = \begin{cases} \dfrac{1}{\|x-y-a\|} & \text{if } x-y\neq a \\ \infty & \text{if } x-y=a \end{cases}$$

f is *bisuperharmonic* and then $M_{s,t} = f(B_s^1, B_t^2)$ is a positive supermartingale. Moreover $(M_{s,t})$ is the increasing limit of $f^n(B_s^1, B_t^2)$ $(f^n = f\wedge n)$ which are continuous supermartingales and therefore $(M_{s,t})$ is of *class R* ([17] the optional stopping theorem applies between two bounded stopping points). (B_t^1, B_t^2) is a brownian motion in $\mathbb{R}^6$ (one can show that its filtration is $(\mathscr{F}_{t,t})_{t\geqslant 0}$).

We set $D_n = \{(x,y)\in\mathbb{R}^6/\|x-y-a\|\leqslant 1/n\}$ and $M_t = M_{t,t} = f(B_t^1, B_t^2)$; then $\{M_t \geqslant n\} = \{(B_t^1, B_t^2)\in D_n\}$; we define σ_n by: $\sigma_n = \inf\{t\geqslant 0/M_t\geqslant n\}$; σ_n is an $(\mathscr{F}_{t,t})$-stopping time and $\sigma_n \uparrow +\infty$ (a.s.) because $\lim_n \uparrow \sigma_n = \inf\{t\geqslant 0/B_t^1 - B_t^2 = a\}$ and $\{(\omega,t)/B_t^1(\omega_1) - B_t^2(\omega_2) = a\}$ is evanescent (as a predictable set which does not contain any graph of stopping time). We set $Z_n = (\sigma_n\wedge n, \sigma_n\wedge n)$ which is a stopping point. We claim that *each M^{Z_n} is a uniformly integrable martingale with respect to $(\mathscr{F}_z^{Z_n})$*. If $z\leqslant z'$, we have $0\leqslant Z_n \dot\wedge z \leqslant Z_n \dot\wedge z' \leqslant Z_n$ and Z_n is bounded. The supermartingale property gives us

$$E(M_{Z_n\dot\wedge z'}/\mathscr{F}_{Z_n\dot\wedge z}) \leqslant M_{Z_n\dot\wedge z} \qquad \text{(a.e.)};$$

but f is *harmonic* in D_n^c shows that $E(M_0) = E(M_{Z_n})$ and therefore $E(M_{Z_n\dot\wedge z'}) = E(M_{Z_n\dot\wedge z})$. Finally $E(M_{z'}^{Z_n}/\mathscr{F}_z^{Z_n}) = M_z^{Z_n}$ (a.e.). In fact, $M_z^{Z_n} = E(M_{Z_n}/\mathscr{F}_{Z_n\dot\wedge z})$ and M^{Z_n} is uniformly integrable (actually bounded by n). $\mathscr{P}_{Z_n}\uparrow(\Omega\times\mathbb{R}_+^2)$ follows from $Z_n = (\sigma_n\wedge n, \sigma_n\wedge n)$, $\sigma_n\uparrow\infty$ and how we obtained the past of a stopping point (X^{+-} and Y^{-+} Section I).

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